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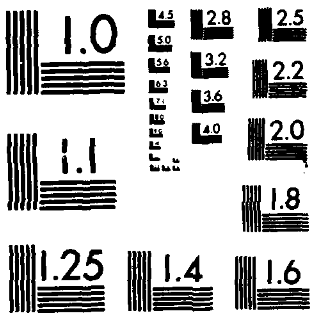

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DYNAMIC REPAIR ALLOCATION FOR A K OUT OF N SYSTEM

MAINTAINED BY DISTINGUISHABLE REPAIRMEN

by

Michael N. Katehakis

and

Costis Melolidakis

NEW YORK 1987

DYNAMIC REPAIR ALLOCATION FOR A K OUT OF N SYSTEM

MAINTAINED BY DISTINGUISHABLE REPAIRMEN.

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ABSTRACT

We consider a K out of N system maintained by R repairmen, where the lifetime of the i^{th} component is an exponentially distributed random variable with parameter μ_i. Repairmen are distinguishable, and the time it takes the r^{th} repairman to repair a failed component is an exponentially distributed random variable with parameter λ_r. Repaired components are as good as new and preemptions are allowed. We show that the policy which assigns the faster repairmen to the most reliable components is optimal with respect to several optimality criteria.

The approach we take in establishing stochastic optimality with respect to the number of functioning components is of wide applicability to different classes of stochastic optimization problems.

1. Introduction. Consider a system that consists of N components. The system is functioning when at least K out of its N components are operating. Components may fail, their lifetimes being exponentially distributed random variables. Failures occur independently and a component may fail even when the system is not functioning. The rate of failure of component i is denoted by μ_i, $i = 1,\dots,N$. The system is maintained by R repairmen. At most one repairman can be assigned to a failed component and a repairman may switch from one failed component to another instantaneously. The time required by the r^{th} repairman to complete the repair of any failed component is considered to be an exponentially distributed random variable with

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parameter λ_r , $r = 1,\ldots,R$. Repaired components are as good as new.

We show that the policy under which the fastest repairman (i.e., the one with the largest repair rate) is assigned to the most reliable failed component (i.e., the one with the smallest failure rate) the next to the fastest repairman is assigned to the next to the most reliable failed component, etc. (FR-MRC policy), possesses the following optimality properties.

It maximizes stochastically the number of working components of the system at any time instant t . Hence, it maximizes both the expected discounted system operation time for all discount rates β , $\beta > 0$ and the average system operation time. Furthermore, it maximizes stochastically the time the system spends in functioning states before a failure occurs, and it minimizes stochastically the time the system is down before it starts operating again.

This problem has been studied for the case in which $R = 1$ in Derman Lieberman and Ross (1980) , where it is shown that the pertinent policy stochastically maximizes the operation time of the system during any time interval. The case in which $R = 1$, failure and repair times are exponentially distributed and the repair rates depend on the components has been considered in Katehakis and Derman (1984) with respect to the average system operation time, and in Katehakis and Melolidakis (1987), with respect to the stochastic optimality of the system nonfunctioning time. See also Smith (1978).

<u>2. Problem formulation.</u> Under the assumptions made, at any time the status of all components is given by a vector $x = (x_1,\ldots,x_N)$ with $x_i = 1$ or 0 according to whether the i^{th} component is functioning or failed. Thus, the set of all possible states is $S = \{0,1\}^N$. For any state $x \in S$, we define states $(\delta_k,x) = (x_1,\ldots,x_{k-1},\delta,x_{k+1},\ldots,x_N)$, $\delta = 0, 1$, $k = 1,\ldots,N$, and we denote the sets of failed and functioning components by $C_0(x)$ and $C_1(x)$,

i.e., $C_0(x) = \{i \mid x_i = 0\}$, $C_1(x) = \{i \mid x_i = 1\}$ respectively.

The cardinality of a set A is denoted by $|A|$. For notational simplicity we define $\|x\| = |C_1(x)|$. Let $S_0 = \{x : \|x\| < K\}$ denote the set of failed states and $S_1 = \{x : \|x\| \geq K\}$ denote the set of functioning states of the system.

We number the components according to their failure rates , i.e., $j \leq i$ if and only if $\mu_j \leq \mu_i$, and the repairmen reversely to their repair rates, i.e., $r \leq s$ if and only if $\lambda_r \geq \lambda_s$.

Let then $q(x)$ be the q^{th} failed component at state x and let $a_q(x)$ be the order of $q(x)$. Let also $\lambda(x) = \sum_{q=1}^{R(x)} \lambda_q$.

The set of repairmen is denoted by $\mathcal{R}$ and $|\mathcal{R}| = R$. Let $R(x)$ denote the maximum number of components that can be under repair when the system is in state x , i.e.,

$$R(x) = \min \{R, |C_0(x)|\} .$$

Let $W(x)$ be the set of all *possible choices* of components on which one may decide to assign repairmen when the system is in state x , i.e.,

$$W(x) = \{ B : B \subset C_0(x) , |B| \leq R(x) \} .$$

For $B \in W(x)$, let $\Lambda(B,x)$ be the set of possible choices of repairmen one may decide to use if he chooses B , i.e.,

$$\Lambda(B,x) = \{ C : C \subset \mathcal{R} , |C| = |B| \} .$$

An assignment of repairmen $C \in \Lambda(B,x)$ to components $B \in W(x)$ is an 1-1 and onto mapping from B to C . Denote by $\Phi(B,C)$ the set of these mappings and let $A(x)$ represent the available actions at state x , i.e.,

$$A(x) = \{ (B,C,\phi) : B \in W(x), C \in \Lambda(B,x), \phi \in \Phi(B,C) \}$$

The problem of the stochastic maximization of the number of functioning components at any time instant t can be formulated as a family of Markovian decision problems in the following way.

Since the state space is finite we can use the device of uniformization (see Lippman (1975) and references given there) which essentially means that we can consider (dummy) transitions back to the same state at such a rate that the sojourn times X_1 , X_2 , ... of the processes are independent exponential random variables with a common rate v . The constant v can be chosen as any number greater than or equal to the sum of the transition rates. The sum $S_n = X_1 + X_2 + \cdots + X_n$ represents the time of the n^{th} transition of the system. Let $n_t = \sup\{n: S_n \leq t\}$, i.e., n_t is is the record of the last transition that occured up to time t . Because of the uniformization, the distributions of S_n and n_t are independent of the particular policy we follow. Given a policy π we define the random variable $N_\pi(t;x)$, where $N_\pi(t;x) = 0,\ldots, N$, represents the number of working components at time t if the state at time 0 was x . Let π_0 denote the FR-MRC policy.

Our aim is to show that

$$P(\ N_{\pi_0}(t;x) \geq k\ ) \geq P(\ N_\pi(t;x) \geq k\ ) \quad \text{for all policies } \pi\ ,\ 0 \leq k \leq N \qquad (1)\ .$$

Let $N_\pi(n;x)$, $n \geq 1$, be the number of working components after the n^{th} transition if the state at time 0 was x . Then, $N_\pi(t;x) = N_\pi(n_t;x)$ and therefore, since

$$P(\ N_\pi(n_t;x) \geq k\ ) = \sum_{n=1}^{\infty} P(\ N_\pi(n;x) \geq k\ )\ P(\ n_t = n\ ) \quad ,\ 0 \leq k \leq N\ .$$

it is sufficient to show that

$$P(\ N_{\pi_0}(n;x) \geq k\ ) \geq P(\ N_\pi(n;x) \geq k\ ) \quad \text{for all policies } \pi\ ,\ \text{all } n \geq 1\ .$$

After uniformization, the relevant family of Markovian decision problems will be the following.

For each n , $n \geq 1$, and for each k , $0 \leq k \leq N$ consider the problem $\Pi_{n,k}$ which is specified by the following elements.

1. State space: the set $\{(x;m) : x \in S , m = 0, 1,\ldots, n \}$.
2. Action sets: the sets $A(x;m) = A(x)$.
3. System dynamics: when the system is in state $(x;m)$ and action $a = (B,C,\phi) \in A(x)$ is chosen , the following transitions are possible :

i) to state $(1_b,x;m-1)$, $b \in B$, with probability $\dfrac{\lambda_{\phi(b)}}{v}$,

ii) to state $(0_\ell,x;m-1)$, $\ell \in C_1(x)$, with probability $\dfrac{\mu_\ell}{v}$,

iii) and to state $(x;m-1)$ with probability $\dfrac{v - \lambda(x,a) - \mu(x)}{v}$, where

$\lambda(x,a) = \Sigma_{b_i \in b} \lambda_{\phi(b_i)}$, $\mu(x) = \Sigma_{\ell \in C_1(x)} \mu_\ell$ and v is any constant greater than or equal to the sum of the transition rates.

iv) Reward structure

$$r(x;m) = \begin{cases} 1 & \text{if } \|x\| \geq k \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that the objective function of $(\Pi_{n,k})$ represents the probability $P(\, N_\pi(n;x) \geq k \,)$, which is to be maximized.

Take $n \geq 1$, and fix k , $0 \leq k \leq N$. Let $v_\pi(x;n)$ denote the value of a policy π in Π_{nk} and let $v(x;n)$ denote the value of an optimal policy. Then $v(.;.)$ is the unique solution to the following system of functional equations;

$$v(x;m+1) = \max_{a \in A(x)} \frac{1}{v} \Big[r(x) + [v - \lambda(x,a) - \mu(x)]v(x;m) + \\ + \Sigma_{b \in B} \lambda_{\phi(b)}\, v(1_{\phi(b)}\, x;m) + \Sigma_{\ell=1}^{N} x_\ell \mu_\ell v(0_\ell x;m) \Big] \qquad (2)$$

where $m = 0,1,\ldots,n-1$.

In the next section we establish that the values of the FR-MRC policy satisfy (2) and thus it is *stochastically optimal with respect to the number of* functioning components in the system at any time instant t .

To show that the FR-MRC policy minimizes (maximizes) stochasticaly the time the system spends in nonfunctioning (functioning) states we *first* need to define the random variables $Y_\pi(x)$, $Z_\pi(x)$, where $Y_\pi(x)$ (respectively $Z_\pi(x)$) denotes the first passage time from a state $x \in S_0$ to the set of functioning states S_1 (respectively from a state $x \in S_1$ to the set of failed states S_0) under a policy π . Related to these two random variables are the *following two families of Markov decision processes:*

The family (Π_n) , $n \geq 1$, that corresponds to $Y_\pi(x)$, is defined by:

1. State space : The set $\{ (x;m) : x \in S , m = 0, 1,..., n \}$.
2. Action sets : The sets $A(x;m) = A(x)$.
3. System dynamics : (I) When the system is in state $(x;m)$, $x \in S_0$ and action $a = (B,C,\phi) \in A(x)$ is chosen , the following transitions are possible :

i) To state $(1_b,x;m-1)$, $b \in B$, with probability $\dfrac{\lambda_\phi(b)}{v}$

ii) To state $(0_\ell,x;m-1)$, $\ell \in C_1(x)$, with probability $\dfrac{\mu_\ell}{v}$

iii) And to state $(x;m-1)$ with probability $\dfrac{v - \lambda(x,a) - \mu(x)}{v}$, where

$\lambda(x,a) = \Sigma_{b_i \in b} \lambda_\phi(b_i)$, $\mu(x) = \Sigma_{\ell \in C_1(x)} \mu_\ell$ and v is any constant greater than or equal to the sum of the transition rates.

(II) When the system is in state $(x;m)$, $x \in S_1$ and action $a \in A(x)$ is chosen, then the next state is $(x;m-1)$ with probability 1.

iv) Reward structure

$$r(x;m) = \begin{cases} 1 & \text{if } x \in S_1 \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

The family (Π_n') , $n \geq 1$, that corresponds to $Z_\pi(x)$, is defined by:
The same state and action sets as in (Π_n) and a transition law described as that of (Π_n) , but with the places of S_0 and S_1 reversed. Similarly, the cost in (Π_n') is defined as in (Π_n) , but with S_0 in the place of S_1 . The objective function is to be maximized in (Π_n) and to be minimized in (Π_n') .

Let $Y_\pi{}^d(x)$ [respectively $Z_\pi{}^d(x)$] represent the number of transitions to absorbtion from an initial state $(x;n)$ in (Π_n) [respectively (Π_n')] . It is easy to see that the objective function in (Π_n) [respectively (Π_n')] represents the $P(Y_\pi{}^d \leq n)$ [respectively $P(Z_\pi{}^d \leq n)$]. To establish the stochastic optimality of π_o for the first passage problem , i.e. that $\forall$ n, $P(Y_{\pi_o}^d \leq n) \geq P(Y_\pi{}^d \leq n)$ [respectively $P(Z_{\pi_o}^d \leq n) \leq P(Z_\pi{}^d \leq n)$] , it is necessary and sufficient to establish the optimality of π_o in (Π_n) and (Π_n') [see Katehakis-Melolidakis (1987)].

__Remark.__ The approach to stochastic optimality using uniformization and Markovian Decision Theory is applicable in more general situations.

3. Optimality properties of the FR-MRC policy.

We first proceed to show that,

__Theorem 1.__ The FR-MRC policy stochastically maximizes the number of functioning components in the system at any time instant t .

__Proof.__ The proof is by establishing the optimality of π_o in the following recursive equations .

$$v(x;m+1) = \max_{a\in A(x)} \frac{1}{v} \Big[r(x) + [v - \lambda(x,a) - \mu(x)]v(x;m) + \\ + \Sigma_{b\in B} \lambda_{\phi(b)} v(1_{\phi(b)} x;m) + \Sigma_{\ell=1}^{N} x_\ell \mu_\ell v(0_\ell x;m) \Big] \qquad (3)_{m+1}$$

$$r(x;m) = \begin{cases} 1 & \text{if } \|x\| \geq k \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

So, we will show that:

$$v(x;m+1) = \frac{1}{v}\Big[r(x) + [v - \lambda(x) - \mu(x)]v(x;m) + $$

$$+ \sum_{q=1}^{R(x)} \lambda_q v(1_{a_q(x)},x;m) + \sum_{\ell=1}^{N} x_\ell \mu_\ell v(0_\ell x;m) \Big] \qquad (4)_{m+1}$$

$$r(x;m) = \begin{cases} 1 & \text{if } \|x\| \geq k \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

The proof is by induction on m .

For $m = 0$ there are four cases.

Case 1: $\|1_i,x\| < k$ for $i \in C_o(x)$. Then, whatever the action a , $v(x;1)$ is independent of a . Hence, $(4)_1$ holds trivially.

Case 2: $\|1_i,x\| = k$ for $i \in C_o(x)$. Then, certainly π_0 maximizes $(3)_1$ among all possible policies ϕ .

Case 3: $\|1_i,x\| = k + 1$ for $i \in C_o(x)$. Then, $v(x;1)$ is independent of the particular policy employed.

Case 4: $\|1_i,x\| > k + 1$ for $i \in C_o(x)$. Then, again any action will guarantee a constant cost.

So, $(4)_1$ is true. Now assume $(4)_m$ is true up to m included. To show that it is true for $m + 1$ also, we need to establish the following lemmas.

<u>Lemma 1.</u> For $j \leq i$, $i,j \in C_o(x)$, the following hold :

a) $v(1_j,x;m) \geq v(1_i,x;m)$ $\qquad (5)_m$

b) $v(1_{aq(1j,x)},1_j,x;m) \geq v(1_{aq(1i,x)},1_i,x;m)$ $\qquad (6)_m$

c) $v(x;m) \leq v(1_j,x;m)$ $\qquad (7)_m$

d) $v(1_{aq(x)},x;m) \leq v(1_{aq(1j,x)},1_j,x;m)$ $\qquad (8)_m$

Proof. The proof is by induction.

For $m = 0$ (5),(6),(7) and (8) are certainly true.

Assume they are true up to $m - 1$ included. We show that they are also true for m. The proof involves the following steps:

a) ($(4)_m$ and $(5)_{m-1}$ and $(6)_{m-1}$ and $(7)_{m-1}$) implies $(5)_m$

b) ($(5)_m$) implies $(6)_m$

c) ($(4)_m$ and $(5)_{m-1}$ and $(7)_{m-1}$ and $(8)_{m-1}$) implies $(7)_m$

d) ($(5)_m$ and $(7)_m$) implies $(8)_m$

More precisely:

a) $(5)_{m-1}$ implies that for $i,j \in C_0(x)$, $j \le i$,

$$[v - \lambda(1_j,x) - \mu_i - \mu(x)][v(1_j,x;m-1) - v(1_i,x;m-1)] \ge 0 \implies$$

$$[v - \lambda(1_j,x) - \mu(x)][v(1_j,x;m-1) - v(1_i,x;m-1)] \ge$$

$$\ge \mu_i[v(1_j,x;m-1) - v(x;m-1)] - \mu_i[v(1_i,x;m-1) - v(x;m-1)] \tag{9}$$

Since $\mu_j \le \mu_i$, $(7)_{m-1}$ and (9) imply that:

$$[v - \lambda(1_j,x) - \mu(x)][v(1_j,x;m-1) - v(1_i,x;m-1)] \ge$$

$$\ge \mu_j[v(1_j,x;m-1) - v(x;m-1)] - \mu_i[v(1_i,x;m-1) - v(x;m-1)]$$

which in turn, after observing that $\|x\| = \|y\| \implies R(x) = R(y) \implies \lambda(x) = \lambda(y)$, leads to :

$$[v - \lambda(1_j,x) - \mu_j - \mu(x)]v(1_j,x;m-1) + \mu_j v(x;m-1)] \ge$$

$$\ge [v - \lambda(1_i,x) - \mu_i - \mu(x)]v(1_i,x;m-1) + \mu_i v(x;m-1)] \tag{10}$$

Also $(6)_{m-1}$ implies that :

$$\Sigma_{q=1}^{R(1_j,x)} \lambda_q v(1_{aq(1_j,x)},1_j,x;m-1) \ge \Sigma_{q=1}^{R(1_i,x)} \lambda_q v(1_{aq(1_i,x)},1_i,x;m-1) \tag{11}$$

and $(5)_{m-1}$ implies that :

$$\Sigma_{\ell\epsilon C_o(x)}\mu_\ell v(0_\ell,1_j,x;m-1) \geq \Sigma_{\ell\epsilon C_o(x)}\mu_\ell v(0_\ell,1_i,x;m-1) \tag{12}$$

By adding (10), (11) and (12) and using $(4)_m$ we conclude that $(5)_m$ holds.

b) To show that $(6)_m$ is true, we have to consider cases.

Case 1: $a_q(1_j,x) < j$, i.e., there are at least q non-functioning components before the j^{th} one. Then, $j \leq i \Longrightarrow a_q(1_i,x) = a_q(1_j,x)$ and $(6)m$ follows from $(5)_m$.

Case 2: $a_q(1_j,x) > j$, i.e., there are less than q non-functioning components before the jth one. Then there are two subcases.

Subcase 2a: $j < a_q(1_j,x) \leq i$. The j^{th} position then counts for $1_i,x$ and hence $a_q(1_i,x) \geq j$. Therefore, repeated use of $(5)_m$ gives :

$$v(1_{aq(1j,x)},1_j,x;m) \geq v(1_{aq(1j,x)},1_{aq(1i,x)},x;m) \geq v(1_i,1_{aq(1i,x)},x;m)$$

which establishes $(6)_m$.

Subcase 2b: $j < i \leq a_q(1_j,x)$. Then $a_q(1_j,x) = a_q(1_i,x)$ and the result follows from $(5)_m$ again.

Hence $(6)_m$ has been established.

c) To show $(7)_m$ we first observe that $(7)_{m-1}$ implies that :

$$[v - \lambda(x) - \mu(x)]v(x;m-1) \leq$$

$$\leq [v - \lambda(x) - \mu_j - \mu(x)]v(1_j,x;m-1) + \mu_j v(x;m-1)] \tag{13}$$

and that,

$$\Sigma_{\ell\epsilon C_o(x)}\mu_\ell v(0_\ell,x;m-1) \leq \Sigma_{\ell\epsilon C_o(x)}\mu_\ell v(0_\ell,1_j,x;m-1) \tag{14}$$

We next consider two cases .

Case 1: $R(x) = R(1_j,x)$, i.e., in both x and $1_j,x$ there are more non-functioning components than available repairmen.

Then, $(8)_{m-1}$ leads to :

$$\sum_{q=1}^{R(x)} \lambda_q v(1_{a_q(x)},x;m-1) \leq \sum_{q=1}^{R(1_j,x)} \lambda_q v(1_{a_q(1_j,x)},1_j,x;m-1) \tag{15}$$

Adding together (13), (14), and (15) and using $(4)_m$ we get :

$v(x;m) \leq v(1_j,x;m)$, which establishes $(7)_m$ for this case.

Case 2: $R(x) = R(1_j,x) + 1$, i.e., in $1_j,x$ there are more repairmen available than failed components.

Again $(7)_{m-1}$ and $(8)_{m-1}$ lead to :

$$\begin{aligned}
&[v - \lambda(x) - \mu_j - \mu(x)]v(x;m-1) + \sum_{q=1}^{R(1_j,x)} \lambda_q v(1_{a_q(x)},x;m-1) + \\
&+ \sum_{\ell \in C_0(x)} \mu_\ell v(0_\ell,x;m-1) \leq \\
&\leq [v - \lambda(x) - \mu_j - \mu(x)]v(1_j,x;m-1) + \sum_{q=1}^{R(1_j,x)} \lambda_q v(1_{a_q(1_j,x)},1_j,x;m-1) + \\
&+ \sum_{\ell \in C_0(x)} \mu_\ell v(0_\ell,1_j,x;m-1)
\end{aligned} \tag{16}$$

Since, in this case, $\lambda(x) = \lambda(1_j,x) + \lambda_{R(x)}$, (16) may be rewritten as :

$$\begin{aligned}
&[v - \lambda(x) - \mu(x)]v(x;m-1) + \sum_{q=1}^{R(1_j,x)} \lambda_q v(1_{a_q(x)},x;m-1) + \lambda_{R(x)} v(1_j,x;m-1) \\
&+ \sum_{\ell \in C_0(x)} \mu_\ell v(0_\ell,x;m-1) \leq \\
&\leq [v - \lambda(1_j,x) - \mu_j - \mu(x)]v(1_j,x;m-1) + \sum_{q=1}^{R(1_j,x)} \lambda_q v(1_{a_q(1_j,x)},1_j,x;m-1) \\
&+ \sum_{\ell \in C_0(x)} \mu_\ell v(0_\ell,1_j,x;m-1) + \mu_j v(x;m-1)
\end{aligned} \tag{17}$$

Now, since for this case $R \geq C_0(x)$, $a_{R(x)}(x)$ is the largest in order component in $C_0(x)$. Hence, $j \leq a_{R(x)}(x)$ which, using $(5)_{m-1}$,implies that

$$\lambda_{R(x)} v(1_{a_{R(x)}}(x), x; m-1) \leq \lambda_{R(x)} v(1_j, x; m-1) \tag{18}$$

But then, using (18) in (17) to strengthen the inequality and using $(4)_m$, we conclude that $(7)_m$ holds for this case also.

d) To show $(8)_m$ we distinguish two cases again.

Case 1: $a_q(1_j,x) < j$. Then $a_q(1_j,x) = a_q(x)$ and $(8)_m$ follows from $(7)_m$

Case 2: $a_q(1_j,x) > j$. Then $a_q(1_j,x)$ is the next to the $a_q(x)$ failed component, which implies that $a_q(x) \geq j$. There are two subcases now:

Subcase 2a: $a_q(x) = j$. Then, $(8)_m$ follows from $(7)_m$.

Subcase 2b: $a_q(x) > j$. Then, using $(5)_m$ and $(7)_m$ we get:

$$v(1_{a_q(x)}, x; m) \leq v(1_{a_q(x)}, 1_{a_q(1_j,x)}, x; m) \leq v(1_j, 1_{a_q(1_j,x)}, x; m)$$

which establishes $(8)_m$.

This finishes the proof of Lemma 1. □

Lemma 2. Among all possible actions that use the first $R(x)$ repairmen, the action determined by π_0 is the best in the optimization of $(3)_{m+1}$.

Proof. It is sufficient to show that whenever $r \geq s$ (r,s representing the order of repairmen) and $j \leq i$ (j,i representing the order of components) then, it is better to assign j to r and i to s than vise versa. The conclusion will then follow after a finite number of permutations of repairmen.

Thus, it suffices to show that if $A \cup \{r\} \cup \{s\}$ is the set of the $R(x)$ first repairmen, $r \leq s$, and if $j \leq i$ then, for any assignment ϕ using all of them, we have:

$$[v-\lambda_r-\lambda_s-\Sigma_{q\in A}\lambda_q-\mu(x)]v(x;m) + \lambda_r v(1_j,x;m) + \lambda_s v(1_i,x;m) + \Sigma_{q\in A}\lambda_q v(1_{\phi-1(q)},x;m)$$

$$+ \Sigma_{\ell\in C_o(x)}\mu_\ell v(0_\ell,x;m) \geq$$

$$[v-\lambda_r-\lambda_s-\Sigma_{q\in A}\lambda_q-\mu(x)]v(x;m) + \lambda_r v(1_i,x;m) + \lambda_s v(1_j,x;m) + \Sigma_{q\in A}\lambda_q v(1_{\phi-1(q)},x;m)$$

$$+ \Sigma_{\ell\in C_o(x)}\mu_\ell v(0_\ell,x;m) \tag{19}$$

But, since $\lambda_r \geq \lambda_s$, (19) is equivalent to $v(1_j,x;m) \geq v(1_i,x;m)$, which is true by Lemma 1(a). □

Now assume that at state x repairmen get assignments sequentially. Then,

<u>Lemma 3.</u> In the optimization problem $(3)_{m+1}$, at each state x we should assign repairmen according to their rates, i.e., first assign the fastest, then the next to the fastest, e.t.c.

<u>Proof.</u> Assume we have already assigned a set of A repairmen, $A \subset R$ with $|A| \leq R(x)$. Then we want to show that if $r \leq s$, $r,s \in R \setminus A$, then:

$$[v - \lambda_r - \Sigma_{q\in A}\lambda_q - \mu(x)]v(x;m) + \lambda_r v(1_i,x;m) + \Sigma_{q\in A}\lambda_q v(1_{\phi-1(q)},x;m) +$$

$$+ \Sigma_{\ell\in C_o(x)}\mu_\ell v(0_\ell,x;m) \geq$$

$$[v - \lambda_s - \Sigma_{q\in A}\lambda_q - \mu(x)]v(x;m) + \lambda_s v(1_i,x;m) + \Sigma_{q\in A}\lambda_q v(1_{\phi-1(q)},x;m) +$$

$$+ \Sigma_{\ell\in C_o(x)}\mu_\ell v(0_\ell,x;m) \tag{20}$$

where i is the failed component to which we decide to assign the next repairman and ϕ is any policy.

Now, since $\lambda_s \leq \lambda_r$, (20) is equivalent to: $v(x;m) \geq v(1_i,x;m)$ which is true (Lemma 1(c)). □

Lemma 4. In the optimization problem $(3)_{m+1}$, at state x we should assign no less than $R(x)$ repairmen to failed components.

Proof. The proof is by induction on $R(x)$. Assume that, according to some policy, we have assigned A repairmen, with $|A| < R(x)$. We show that it is then better to assign one more. Take $r \in \mathcal{R} \setminus A$. We want to show that,

$$[v - \lambda_r - \Sigma_{q\in A}\lambda_q - \mu(x)]v(x;m) + \lambda_r v(1_i,x;m) + \Sigma_{q\in A}\lambda_q v(1_{\phi-1(q)},x;m) +$$

$$+ \Sigma_{\ell\in C_o(x)}\mu_\ell v(0_\ell,x;m) \geq$$

$$\geq [v - \Sigma_{q\in A}\lambda_q - \mu(x)]v(x;m) + \Sigma_{q\in A}\lambda_q v(1_{\phi-1(q)},x;m) + \Sigma_{\ell\in C_o(x)}\mu_\ell v(0_\ell,x;m) \tag{21}$$

where i is any failed component that does not get assigned a repairman from A. But (21) is equivalent to $v(x;m) \leq v(1_i,x;m)$ which is true (Lemma 1(c)).□

Lemmas (3), (4) and (5) are enough to complete the induction step in the proof of Theorem 1. Indeed, in the optimization of $(3)_{m+1}$ at state x we should use the first $R(x)$ repairmen (lemmas 3 and 4 combined) and among all policies that use these first $R(x)$, π_o is the best (lemma 2). Hence, the inductive step is complete and Theorem 1 has been proved.□

Corollary 1. The FR-MRC policy maximizes both the total expected discounted operation time of the system and the availability of the system.

Proof. Notice that since the total discounted operation time of the system is given by $\int_0^\infty e^{-\beta t} P(N_\pi(t;x) \geq K)\, dt$ and since the average operation time of the system is given by $\lim_{T\to\infty} \frac{1}{T} \int_0^T P(N_\pi(t;x) \geq K)\, dt$, the stochastic optimality of π_o implies its optimality with respect to the other two criteria.

Theorem 2. The FR-MRC policy stochastically minimizes the time the system spends in non-functioning states until it starts operating again. Moreover, π_o stochastically maximizes the operation time of the system untill failure.

Proof. According to the discussion in section 2, it is sufficient to prove the optimality of π_0 for the two families (Π_n) and (Π_n'). Now, it is very easy to check that all the arguments in the proof of Theorem 1 go through for the (Π_n) problem also, establishing thus the optimality of π_0. The same arguments work for (Π_n') also, but with the sign of all numbered inequalities reversed. □

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8